Triangulated persistence category (TPC) in symplectic geometry

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Motivation

• Roughly speaking, Kontsevich's homological mirror symmetry (HMS) (1994) claims that

$$D\mathrm{Fuk}(X) \simeq D\mathrm{Coh}(X^\vee)$$

where X is a symplectic manifold and $X^\vee$ is its mirror.

• The A-side $D\mathrm{Fuk}(X)$ and the B-side $D\mathrm{Coh}(X^\vee)$ share some common algebraic structures, for instance, the triangulated structure (cones, distinguished triangles, etc.)

• Recent work by Biran-Cornea (2014) reinterprets the triangulated structure on $D\mathrm{Fuk}(X)$ in terms of their Lagrangian cobordism theory. In particular, one can define the cone decomposition (iterated cones) and count the number of triangles in this decomposition.

• Recent work by Dimitrov-Haiden-Katzarkov-Kontsevich (2014) defines a complexity in $D\mathrm{Coh}(X^\vee)$ when it is split-generated. Also Orlov has some work on the dimension of $D\mathrm{Coh}(\cdot)$ (2008).

Motivation (cont.)

• However, the A-side $D\mathrm{Fuk}(X)$ seems having richer structures than the B-side $D\mathrm{Coh}(X^\vee)$. For instance, it can be studied from a quantitative perspective, since the building block

Lagrangian Floer complex $\mathrm{CF}(L, L')$ is a filtered vector space

where the filtration comes from the symplectic action functional (defined on some path space).

• Quantitative measurements on $\mathrm{CF}(L, L')$ include spectral invariants (Oh, Viterbo), torsion exponents (FOOO), boundary depth (Usher), barcode (Usher-Z.), partially symplectic quasi-state (Polterovich-Entov) etc.

• Recent work by Biran-Cornea-Shelukhin (2021) defines the concept "shadow", a positive real number, for each cone decomposition when we study $D\mathrm{Fuk}(X)$ from the perspective of Lagrangian cobordism theory.

Born of TPC

Inspired by Biran-Cornea-Shelukhin's work (2021), one can combine the filtration structure and triangulated structure in some way, so that one can easily extract numerical data from $D\mathrm{Fuk}(X)$. One approach is called

triangulated persistence category (TPC)

which is, more precisely,

a persistence category + triangulated structure.

Remark

The word "persistence" is from persistent homology or persistence module, a theoretical foundation of the newly-developed field - topological data analysis (TDA). Background on TDA is *not* required for the rest of this talk.

Persistence category

Definition (Biran-Cornea-Z. 2021)

A **persistence category** (PC) $\mathscr{C}$ is a (classical) category enriched by persistence modules. Explicitly, if for any $A, B \in \mathrm{Ob}(\mathscr{C})$, there exists a functor $E_{A,B} : (\mathbb{R}, \leq) \to \mathrm{Vect}_{\mathbf{k}}$ satisfying

(i) the hom-set in $\mathscr{C}$ is $\mathrm{Hom}_{\mathscr{C}}(A, B) = \{(f, r) \,|\, f \in E_{A,B}(r)\}$. For later use, denote $\mathrm{Mor}^r(A, B) := E_{A,B}(r)$;

(ii) for $r \leq r'$ and $s \leq s'$, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}^r(A,B) \times \mathrm{Mor}^s(B,C) & \xrightarrow{\circ_{(r,s)}} & \mathrm{Mor}^{r+s}(A,C) \\ \Big\downarrow {\scriptstyle E_{A,B}(i_{r,r'}) \times E_{B,C}(i_{s,s'})} & & \Big\downarrow {\scriptstyle E_{A,C}(i_{r+s,r'+s'})} \\ \mathrm{Mor}^{r'}(A,B) \times \mathrm{Mor}^{s'}(B,C) & \xrightarrow{\circ_{(r',s')}} & \mathrm{Mor}^{r'+s'}(A,C) \end{array}$$

The functor $E_{A,B}$ is called a persistence $\mathbf{k}$-module and $i_{r,s} = E_{A,B}(i_{r,s})$ for $r \leq s$ is called a structure map.

Persistence category (cont.)

• Observe that a PC $\mathscr{C}$ is not necessarily additive, that is, $\mathrm{Hom}_{\mathscr{C}}$ is not always abelian. Here are two ways that help us transfer back to abelian categories.

(a) Denote by $\mathscr{C}_0$ the restriction category with $\mathrm{Hom}_{\mathscr{C}_0} = \mathrm{Mor}^0$.

(b) Denote by $\mathscr{C}_\infty$ the limit category with $\mathrm{Hom}_{\mathscr{C}_\infty} = \varinjlim_r \mathrm{Mor}^r$.

• One can define persistence functors between two PCs, persistence natural transformations, etc. There will be no surprise, only conventions!

• An important but auxiliary data for a PC $\mathscr{C}$ is a **shift functor** $\Sigma = (\{\Sigma^r\}_{r\in\mathbb{R}}, \{\theta_{s,t}\}_{s,t})$, an $\mathbb{R}$-parametrized family of persistence functors on $\mathscr{C}$ with persistence natural transformations satisfying certain conditions. For instance, evaluated at any object $A \in \mathrm{Ob}(\mathscr{C})$,

$$\theta_{r,0}(A) \in \mathrm{Mor}^{-r}(\Sigma^r A, \Sigma^0 A) = \mathrm{Mor}^{-r}(\Sigma^r A, A).$$

When $r \geq 0$, map $i_{-r,0}(\theta_{r,0}(A)) \in \mathrm{Mor}^{0}(\Sigma^r A, A)$ is important.

Acyclic objects

• In a PC with a shift functor, different from the classical category, elements can be considered **up to shift/approximation**. One views $i_{-r,0}(\theta_{r,0}(A))$ as an r-approximation of the identity map e_A.

Definition

An object $A \in \mathrm{Ob}(\mathscr{C})$ is called r-**acyclic** if $i_{-r,0}(\theta_{r,0}(A)) = 0$.

Recall that an object A is 0 in any category if and only if $e_A = 0$.

• The object A is r-acyclic if and only if $i_{r,0}(e_A) = 0$ (Exercise).

• There are various equivalent names for acyclic objects (depending on the context) such as torsion element, homological non-essential element, etc. The most successful applications of the persistence module theory to symplectic geometry in recent years almost all come from the study of acyclic objects (*very biased opinion*!)

Adding the triangulated structure to a PC (with a shift functor) will lead to something interesting. Let us highlight three concepts that can be derived from a classical triangulated structure.

(1) **Cone decomposition**. Fix a subset of objects $\mathscr{X} := (X_0, X_1, ..., X_n)$. A cone decomposition of any object A with respect to $\mathscr{X}$ consists of a family of triangles $(\Delta_1, \cdots, \Delta_n)$ in $\mathscr{C}$ as follows,

$$\begin{cases} \Delta_1 : & X_0 \to Y_1 \to X_1 \to TX_0 \\ \Delta_2 : & Y_1 \to Y_2 \to X_2 \to TY_1 \\ & \vdots \\ \Delta_{n-1} : & Y_{n-2} \to Y_{n-1} \to X_{n-1} \to TY_{n-2} \\ \Delta_n : & Y_{n-1} \to A \to X_n \to TY_{n-1}. \end{cases}$$

Cone decomposition can be concatenated (or called refinement).

(2) **Complexity**. Suppose $\mathscr{C}$ is split-generated by a single object E. Consider, in a cone decomposition as above,

$$X_i = T^{m_i} E \quad \text{for some } m_i \in \mathbb{Z}.$$

Define (due to DHKK 2014), for $A \in \mathrm{Ob}(\mathscr{C})$,

$$\delta_t(E, A) = \inf \left\{ \sum_{i=1}^{n} e^{-m_i t} \;\middle|\; \begin{array}{l} \text{the cone decomposition of } A \\ \text{with respect to translations of } E \end{array} \right\}.$$

(3) **Grothendieck group**. Let $\mathscr{C}$ be a triangulated category, its K-group is defined, as an abelian group, by

$$K(\mathscr{C}) = \left\langle A \in \mathrm{Ob}(\mathscr{C}) \;\middle|\; \begin{array}{l} A = B + C \text{ if and only if} \\ \exists\, B \to A \to C \to TB \text{ in } \mathscr{C} \end{array} \right\rangle.$$

One should view $K(\mathscr{C})$ as a linear approximation of $\mathscr{C}$.

Main results

Theorem (Biran-Cornea-Z. 2021)

Let (X, ω) be an exact symplectic manifold. Then the derived Fukaya category $\mathrm{DFuk}(X)$ is a TPC.

For simplicity, consider dg-categories. Recall the following steps that produce a triangulated structure (by Bondal-Kapranov 1990):

$$\text{dg-category} \rightsquigarrow \text{pre-triangulated completion} \rightsquigarrow \text{homotopy category}.$$

We can enhance everything to be filtered:

$$\begin{array}{c}\text{filtered}\\ \text{dg-category}\end{array} \rightsquigarrow \begin{array}{c}\text{filtered}\\ \text{pre-triangulated completion}\end{array} \rightsquigarrow \begin{array}{c}\text{filtered}\\ \text{homotopy category}\end{array}.$$

In particular, the hom-set of a filtered dg-category is a **filtered chain complex**. The resulting filtered homotopy category is a TPC (Biran-Cornea-Z. 2021).

Main results (cont.)

Theorem (Biran-Cornea-Z. 2021)

For any TPC $\mathscr{C}$, we can define a non-trivial pseudo-metric called fragmentation pseudo-metric d_{frag}, **with values in $\mathbb{R}_{\geq 0}$**, *on* $\text{Ob}(\mathscr{C})$.

• This d_{frag} is a purely algebraic generalization of the "fragmentation shadow metric" defined by Biran-Cornea-Shelukhin (2021) via Lagrangian cobordism theory.

• This d_{frag} is defined via a quantity in $\mathbb{R}_{\geq 0}$ associated to any cone decomposition and the difficulty is to show this quantity behaves in a sub-additivity way with respect to the concatenation/refinement.

• In terms of d_{frag}, one can study the dynamical property of endomorphisms F on $\mathscr{C}$. For instance, for any "generator" A of $\mathscr{C}$, consider $\lim_{n\to\infty} \frac{d_{\text{frag}}(F^n A, A)}{n}$.

• The estimation of d_{frag} is possible but computing the exact value of d_{frag} is almost never possible (even when $\mathscr{C}$ is very concrete).

Main results (cont.)

• Denote by $FK_{\mathbf{k}}$ the category of filtered chain complexes over $\mathbf{k}$. Its filtered homotopy category, denoted by $H_0(FK_{\mathbf{k}})$, is a TPC.

• A result by Usher-Z. (2016) provides a complete classification of objects in $H_0(FK_{\mathbf{k}})$. Explicitly,

$$X = \bigoplus_{a\in\mathbb{R}, m\in\mathbb{Z}} E_1(a)[m] \oplus \bigoplus_{b\leq c\in\mathbb{R}, m'\in\mathbb{Z}} E_2(b,c)[m']$$

where $E_1(a) = (\cdots \to \langle x\rangle \to \cdots)$ and $E_2(b,c) = (\cdots \to \langle z\rangle \xrightarrow{\partial} \langle y\rangle_{\mathbf{k}} \to \cdots)$ where filtrations of x, y, z are a, b, c (spectrum) respectively.

Theorem (Biran-Cornea-Z. 2022)

In the TPC $H_0(FK_{\mathbf{k}})$, the (modified) DHKK-complexity is

$$\delta_t(E_1(0), X) = \sum_{deg\ k} \sum_i \left(e^{-a_{k_i} t} - e^{-b_{k_i} t}\right) = magnitude\ of\ H_*(X).$$

Here, (persistence) magnitude is from Govc-Hepworth (2021).

Main results (cont.)

Theorem (Biran-Cornea-Z. 2022)

For any TPC $\mathscr{C}$, there exists a pairing, a bilinear (but not necessarily symmetry) map

$$\kappa : K(\mathscr{C}) \otimes K(\mathscr{C}) \to \Lambda_{\mathrm{P}}$$

where $K(\mathscr{C})$ is the K-group of the restriction category $\mathscr{C}_0$ and Λ_{P} is the Novikov polynomial ring $\{\sum_{finite} n_r t^r \mid n_r \in \mathbb{Z},\ r \in \mathbb{R}\}$.

This pairing κ is a generalization of the classical Euler pairing, but the target Λ_{P} is novel (and somewhat mysterious).

Corollary

Let X be a plumbing of two copies of T^*S^1. Then there exists $M \in K(\mathrm{DFuk}(X))$, as a K-class, that can not *be represented by any embedded closed Lagrangian of X.*

Main results (cont.)

- Basic fact: for a TPC $\mathscr{C}$, the restriction of $\mathscr{C}$ on acyclic objects is also a TPC, denoted by $\mathscr{A}\mathscr{C}$.

- For any TPC $\mathscr{C}$, the K-group $K(\mathscr{C})$ can always be viewed as a Λ_{P}-module, explicitly by $t^r \cdot [A] := [\Sigma^r A]$ where Σ^r is the shift functor from the underlying PC structure of $\mathscr{C}$.

- Denote by $\mathscr{T}PC$ the category of TPCs and $\mathscr{M}_{\Lambda_{\mathrm{P}}}$ the category of Λ_{P}-modules. One can consider the following three functors from $\mathscr{T}PC$ to $\mathscr{M}_{\Lambda_{\mathrm{P}}}$:

$$K(\mathscr{C}) := K(\mathscr{C}_0),\ K\mathscr{A}(\mathscr{C}) := K(\mathscr{A}\mathscr{C}_0),\ K_\infty(\mathscr{C}) := K(\mathscr{C}_\infty).$$

Theorem (Biran-Cornea-Z. 2022)

There exists a functor $\mathrm{Tor}K : \mathscr{T}PC \to \mathscr{M}_{\Lambda_{\mathrm{P}}}$ *such that the following sequence of functors is exact,*

$$0 \to \mathrm{Tor}K \to K\mathscr{A} \to K \to K_\infty \to 0.$$

TPC

Definition

A **triangulated persistence category** (TPC) is a triple $(\mathscr{C}, \Sigma, T)$ satisfying the following conditions.

(i) $\mathscr{C}$ is a persistence category endowed with a shift functor Σ.

(ii) $\mathscr{C}_0$ is triangulated where T is the translation autoequivalence.

(iii) Functors Σ and T commute.

(iv) For $r \geq 0$ and $A \in \text{Ob}(\mathscr{C})$, the map $i_{-r,0}(\theta_{r,0}(A)) : \Sigma^r A \to A$ completes into a triangle in $\mathscr{C}_0$ as follows,

$$\Sigma^r A \xrightarrow{i_{-r,0}(\theta_{r,0}(A))} A \to K \to T\Sigma^r A$$

such that K is r-acyclic.

One should view (iv) as an r-approximation of the standard axiom $A \xrightarrow{e_A} A \to 0 \to TA$.

Keys to proofs

- In a TPC, one can define a weighted triangle in the form of

$$A \to B \to C \to \Sigma^{-r} TA.$$

This is a new family of triangles in $\mathscr{C}_0$ and it is "r-closed" to a standard triangle. From this, one can derive weighted octahedral axiom.

- One can compute $K(H_0(FK_{\mathbf{k}}))$. At present, this is the only case that we can give a precise answer (Biran-Cornea-Z. 2022).

$$\lambda : K(H_0(FK_{\mathbf{k}}^{\text{f.g.}})) \simeq \Lambda_P$$

where f.g. means every filtered chain complex is in total of finite rank, and the isomorphism λ is an isomorphism of rings.

- The isomorphism λ above can be described explicitly by $\lambda : [E_1(a)] \to t^a$ (hence, $[E_2(b,c)] = t^b - t^c$).

Keys to proofs (cont.)

• The pairing κ can be constructed by the following recipe (first on the level of $\mathrm{Ob}(\mathscr{C})$). For any $A, B \in \mathrm{Ob}(\mathscr{C})$, consider

$$A, B \rightsquigarrow \mathrm{Hom}_{\mathscr{C}}(A, T^i B) \rightsquigarrow X_{A, T^i B} \in \mathrm{Ob}(FK_{\mathbf{k}}) \rightsquigarrow \lambda(X_{A, T^i B}),$$

then define

$$\kappa([A], [B]) := \sum_i (-1)^i \lambda(X_{A, T^i B}) \in \Lambda_{\mathrm{P}}.$$

The difficulty is to show κ is well-defined on the level of K-group.

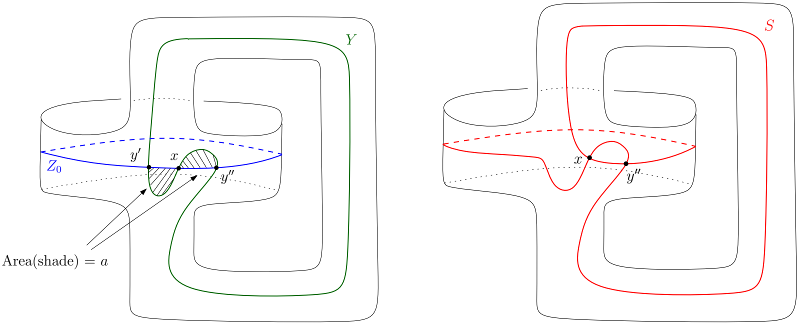
• When $\mathscr{C}$ satisfies certain duality (say, Calabi-Yau duality), then for any $a, b \in K(\mathscr{C})$, one obtains the following skew-symmetry,

$$\kappa(a, b)(t) = \pm \kappa(b, a)(t^{-1}).$$

This implies that in $D\mathrm{Fuk}(X)$, we have $\kappa([L], [L]) = \chi(L)$, which is in particular a constant!

Keys to proofs (cont.)

- Here is the explicit picture for Corollary above.



where the immersed Lagrangian S is constructed by doing a surgery at the intersection point y'. Denote by the M_S in $D\mathrm{Fuk}(X)$ the module represented by S, then one computes that

$$\kappa([M_S],[M_S]) = t^{-a} - t^{a}.$$

K-theoretical measurements

The quantitative nature of a TPC $\mathscr{C}$ allows one to study its K-group $K(\mathscr{C})$ from a quantitative perspective.

- When $\mathscr{C} = H_0(FK_{\mathbf{k}})$, we start from the following observation (Biran-Cornea-Z. 2022),

$$\theta : \Lambda_{\mathrm{P}} \simeq LC_B$$

where $LC_B = \left\{\sum_{\text{finite}} n_I 1_I \,|\, I \subset \mathbb{R} \text{ interval}, n_I \in \mathbb{Z}\right\}$ and 1_I is the indicator of I. Importantly θ, defined by $t^a \mapsto 1_{[a,\infty)}$, is an isomorphism of rings, **but the ring structure on LC_B is peculiar**. Then one can apply various operations on functions to elements σ in LC_B. For instance,

$$\sigma \mapsto \int_{\mathbb{R}} (\sigma - \sigma(\infty) 1_{[0,\infty)}) d\mu.$$

This is called the length of σ and $\sigma(\infty)$ is called the Euler characteristic of σ.

K-theoretical measurements (cont.)

• For a general TPC $\mathscr{C}$ where $\mathrm{Ob}(\mathscr{C})$ is endowed with a pseudo-metric $d_{\mathscr{C}}$, one can define

$$d_{K(\mathscr{C})}(a,b) := \inf \left\{ d_{\mathscr{C}}(A,B) \,\middle|\, [A]=a,\ [B]=b \right\}.$$

So far, this approach is embarrassing since for most examples that we know, for instance d_{frag}, the induced pseudo-metric $d_{K(\mathscr{C})}$ is trivial.

• The deep reason is that when passing to the K-group, one can *not* distinguish "small torsion" with "big torsion". Here is an example. Again, for $\mathscr{C} = H_0(FK_{\mathbf{k}})$, we have

$$[E_2(0,100)] = [E_2(0,1)] + \cdots + [E_2(99,100)]$$

in its K-group.

• One can use filtration shifts of autoequivalences of $\mathscr{C}$ to measure the distance between objects in $\mathscr{C}$ and then descent to $K(\mathscr{C})$.

Outlook

- Categorical dynamics on $K(\mathscr{C})$ (TPC + DHKK 2014).

- Quantitative refinement of homological mirror symmetry (??)

- Filtered derived functors/derived category (part of these has been written up)

- Other algebraic structures that combine the filtration structure and the triangulated structure (PTC?).

THANK YOU!